The Electronic Journal of Mathematics and Technology, Volume 19, Number 1, ISSN 1933-2823

Topology of Quartic Loci in 2D and 3D Inspired by A College Entrance Exam

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Abstract

We explore with technological tools on some problems that originated from a college practice entrance question (see [12]). The problems involve an affine transformation whose image is a locus based on lines passing through a fixed point (see Figure 1(a) or 1(b)), which has been explored in ([17], [18], [19], [20]) by using parametric equations. We are interested in the topological structures for the shapes of image curves or surfaces when the scaling factor s vary. We found the value for s which affects the topological structures for the loci in 2D first, and we extend the results to 3D based on the arguments in 2D. Finally, we include an exploration on the topological structure for the locus ellipsoid when scaling factor s and the fixed point A approach to infinity.

1 A College Entrance Exam Practice Problem from China

We explore a problem that originated from a college practice entrance problem (see [12]).

Explorations: *We are given a fixed circle and a fixed point A in the interior of the circle $(x-a)^2+(y-b)^2 = r^2$. A line passes through A and intersects the circle at C and D respectively, and the point E is the midpoint of CD. Find the locus of E when C moves along the circle.*

This problem has been discussed using parametric equation approaches in 2D and 3D, see ([17], [18], [19], [20]). In this paper, we will use the implicit equation of the locus and assume the fixed $A = (a_1, a_2)$ to be outside the circle or the ellipse. We are given a fixed point $A = (a_1, a_2)$ and the lines passing through this fixed point to intersect a conic $c : x^2 + qy^2 = 1$ at C and D ($q > 0$). The point E lies on CD and satisfies

$$\overrightarrow{ED} = s \cdot \overrightarrow{CD}, \tag{1}$$

where s is a given real number. For convenience, we sometimes write this as

$$E = sC + (1 - s)\, D. \tag{2}$$

Since D is one of the points of intersection between the line passing through A and the conic c, sometimes we call D as the antipodal point of the point C. We remind the readers that a generalized plane quartic curve is a bivariate quartic equation (see [16]). Our aim is to study the topology of the obtained parametrized quartic curve $\mathcal{C}(a_1, a_2, q, s; x, y)$, and determine the value(s) of s which will change the topological structures for the quartic curve $\mathcal{C}(a_1, a_2, q, s; x, y)$, and we extend the results to 3D accordingly.

2 Preliminary

A topological space is a non-empty set together with a topological structure that is given by the open sets of the topology. A non-empty set X can be equipped with different topological structures, and any subset $A \subset X$ can be endowed with an inherited topological structure, namely, the *relative topology*: the open sets in the relative topology on A are exactly the subsets of the form $A \cap O$ for some open sets O of X.

For the spaces $\mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$, the usual topological structure is given by a base of open balls, which is defined by the metric space structure of $\mathbb{R}^n$. Recall that the *metric function* $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}$, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ makes $(\mathbb{R}^n, d)$ a *metric space*. This metric gives rise to open balls $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid d(x, y) < \varepsilon\}$, where ε is a positive real number, which are the basic open sets for the usual topology on $\mathbb{R}^n$.

Thus, any curve or any hyper-surface in $\mathbb{R}^n$ can be viewed as topological spaces with their respective relative topologies, acquired from the space in which they are embedded. For the cases we are interested in, the environment spaces will be simply $\mathbb{R}^2$ or $\mathbb{R}^3$; thus, we can view these curves and surfaces as relative topological spaces on which we can apply topological tools that are introduced in what follows.

Now, let us recall a fundamental concept in topology, namely that of *continuity*. Let (X, τ_X) and (Y, τ_Y) be two topological spaces, with τ_X and τ_Y their respective topologies. A function $f : X \longrightarrow Y$ is *continuous* if for any open set $O \subset Y$ it holds that $f^{-1}(O) \subset X$ is open in X. Here, the notation $f^{-1}(U)$ denotes the inverse image, in X, of $U \subset Y$.

When f is bijection, then it has an inverse $f^{-1} : Y \longrightarrow X$ defined by: for any $y \in Y$, $f^{-1}(y) = x \in X$ if and only if $f(x) = y$. We say that f is a *homeomorphism* if it is a continuous bijection and with f^{-1} also being continuous. In this situation, X and Y are said to be *homeomorphic* topological spaces.

Homeomorphisms between topological spaces are also called *continuous transformations*, or *topological transformations*. Intuitively, these transformations can be thought of as functions that transform points of one space, that are arbitrarily close to each other, onto points of another space that are also arbitrarily close. Spaces that are related in this way are said to be *topologically equivalent*. In a technical sense, we have to be careful with the notion of closeness since it only makes sense when we can measure distances in a space, so we need to have a metric structure. Nevertheless, that is what we meant by “intuitively”. It is intuitively evident that all simple closed curves in the plane and all closed polygons are topologically equivalent to a circle. Similarly, all closed cylinders, closed cones, convex polyhedra, and other simple closed surfaces are topologically equivalent to a sphere.

For the problem described in the“**Exploration**”, now we are interested in a fixed point A outside of a given conic. For example, the loci in red are shown in Figures 1(a) and (b), when the fixed point A is outside a circle, $x^2 + y^2 = r^2$. More specifically, we encourage the readers to use their favorite DGS (Dynamic Geometry System) tools to explore the followings:

1. When $A = (1.598024, 0.6045678)$, $s = 0.25$ and $r = 2$, we provide such case in the following screenshot using [2] in Figure 1(a).

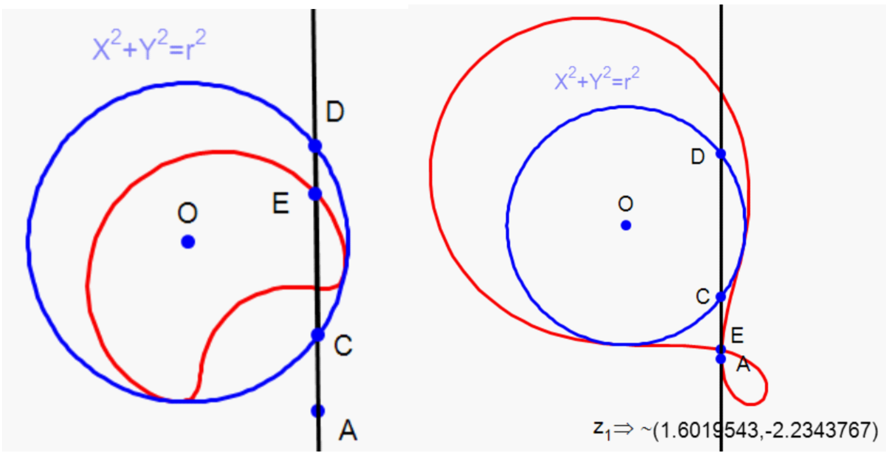

Figure 1(a) (left). The fixed point A is outside the circle and $s = 0.25$
Figure 1(b) (right). The fixed point A is outside the circle and $s = 1.43$

2. If $A = (1.6019543, -2.2343767)$ is outside the circle, and $s = 1.43$, the locus is shown in Figure 1(b).

Among other techniques, we used the automated GeoGebra Discovery (see [3]) to obtain the accurate locus by computing the locus equation for the investigated geometric setups automatically. The following table explains how a possible construction protocol can be given by the user to get a quartic polynomial equation for the case $s = -1.5$, $A = (2, 1)$, if the unit circle is used. Recent versions of GeoGebra Discovery allow the use of the Dilate tool in a symbolic environment, therefore it is expected to get the output locus equation very quickly. As of today, version 2024Jun29 can compute and plot the corresponding quartic curves at about 9.9 frames per second.

No.	Name	Icon	Description	Definition	Value
1	Number s				$s = -1.5$
2	Point A				$A = (2, 1)$
3	Point O		Intersection of xAxis and yAxis	$Intersect\,(xAxis, yAxis)$	$O = (0, 0)$
4	Point U		Point on xAxis	$Point\,(xAxis)$	$U = (1, 0)$
5	Circle c		Circle through U with center O	$Circle\,(O, U)$	$c : x^2 + y^2 = 1$
6	Point D		Point on c	$Point\,(c)$	$D = (0.53, 0.85)$
7	Line l		Line A, D	$Line\,(A, D)$	$l : 0.15x - 1.47y = -1.16$
8	Point C		Intersection of c and l	$Intersect\,(c, \ell, 2)$	$C = (-0.7, 0.71)$
9	Point E		C dilated by factor s from D	$Dilate\,(C, s, D)$	$E = (2.39, 1.04)$
10	Implicit Curve eq1		LocusEquation(E, D)	$LocusEquation\,(E, D)$	$eq1 : x^4 - 4x^3 + 2x^2y^2 - 2x^2y + 4x^2 - 4xy^2 - 60xy + 64x + y^4 - 2y^3 + 49y^2 + 32y = 80$

2.1 The special case of a circle

We consider the circle when $a_1 = 0, a_2$ is any real number, $q = 1$, and by using symmetry, the fixed point $A = (0, a_2)$ on the y-axis can be assumed. For example, we use [4] to demonstrate such a scenario in the following Figure 1(c).

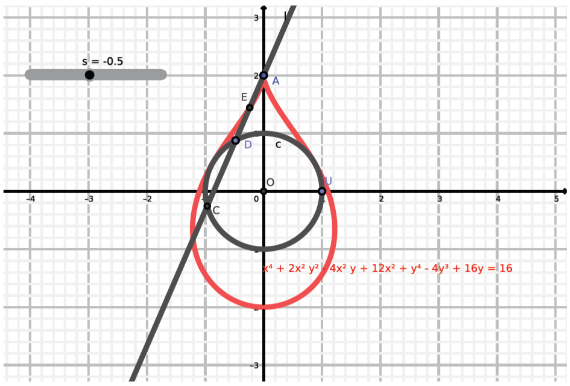

Figure 1(c). Locus in red

Intuitively, we see that the topological structures for the loci, shown in red color, in Figures 1(b) and (c) are different. The Figure 1(b) looks like a figure 8 with one point of self-intersection, and Figure 1(c) is not and has a cusp. Therefore, we say that the Figure 1(b) is not topologically equivalent to the Figure 1(c). In this exploration, we study the topology of the obtained family of parametrized quartic curves, which is denoted by $\mathcal{C}(a_1, a_2, q, s; x, y)$. We shall prove that the topology is changing at

$$s = \frac{1 \pm \sqrt{a_1^2 + qa_2^2}}{2}.$$

We proceed with the following steps:

1. We look for the algebraic equations for points on a circle, collinearity and dilation.

2. We use elimination, by choosing a sensible polynomial factor of the generator polynomial p of the elimination ideal, so a parametrized curve $\mathcal{C}$ can be obtained.

3. The solutions of the equation system $\{\mathcal{C} = 0, \frac{d\mathcal{C}}{dx} = 0, \frac{d\mathcal{C}}{dy} = 0\}$ can help to find the critical points of all possible curves.

2.2 System of equations for the special case of a circle

We consider the system of equations for $\mathcal{C}(a_1 = 0, a_2, q = 1, s; x, y)$, where a_2 is any real number:

$$\begin{aligned}
&e_1 := c_1^2 + c_2^2 = 1, \quad e_2 := d_1^2 + d_2^2 = 1,\\
&e_3 := \det \begin{pmatrix} c_1 & c_2 & 1 \\ d_1 & d_2 & 1 \\ 0 & a_2 & 1 \end{pmatrix} = 0,\\
&e_4 := x - d_1 = s \cdot (c_1 - d_1),\\
&e_5 := y - d_2 = s \cdot (c_2 - d_2),\\
&I := \langle e_1, e_2, \ldots, e_5 \rangle \cap \mathbb{Q}[a_2, s, x, y], \quad \langle p \rangle := I, \quad \mathcal{C} \mid p,\\
&S := \{\mathcal{C} = 0, \frac{d\mathcal{C}}{dx} = 0, \frac{d\mathcal{C}}{dy} = 0\}.
\end{aligned}$$

The following **Giac code** (see [5]) solves the system:

```
e1:=d1^2+d2^2=1
e2:=c1^2+c2^2=1
e3:=det([c1,c2,1],[d1,d2,1],[0,a2,1])
e4:=x-d1=s*(c1-d1)
e5:=y-d2=s*(c2-d2)
I:=eliminate([e1,e2,e3,e4,e5],[c1,c2,d1,d2])
p:=I[0]
c:=(factor(p))[2]
dcx:=diff(c,x)
dcy:=diff(c,y)
S:=solve([c,dcx,dcy],[x,y,s])
```

In view of the intermediate step, when computing the principal ideal $I = \langle p \rangle$, is of the form

$$p = \left(x^2 + y^2 - 1\right) \left(\begin{array}{c} 4a_2^2x^2s^2 - 4a_2^2x^2s + a_2^2x^2 - 4a_2^2s^2 + 4a_2^2s + a_2^2y^2 - a_2^2 \\ -2a_2x^2y + 8a_2s^2y - 8a_2sy - 2a_2y^3 + 2a_2y \\ +x^4 - 4x^2s^2 + 4x^2s + 2x^2y^2 - x^2 - 4s^2y^2 + 4sy^2 + y^4 - y^2 \end{array} \right), \qquad (3)$$

we note that the first factor corresponds to a **degenerate case**. Therefore, we only need the second factor in the next intermediate step: `c:=(factor(p))[2]`.

Solution (with Giac), computed via Gröbner bases and other techniques:

$$
\begin{aligned}
(x, y, s) = \{ & \\
& (0, a_2, 0), \\
& (0, a_2, 1), \\
& \left(\pm\sqrt{-y^2 + a_2 y}, y, \frac{1}{2}\right), \\
& \left(0, a_2, \frac{1 \pm a_2}{2}\right), \\
& \left(0, a_2, \frac{1}{2}\right), \\
& (0, a_2, s) \\
\}. &
\end{aligned}
$$

Geometrically, only the solution $x = 0$ makes sense. Assuming a special case $a_2 = 2$, the candidates for s are: $0, 1, \frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$. We shall see when $s = \frac{3}{2}$ or $-\frac{1}{2}$ (which produces the same locus), is the boundary where the topology of the locus changes. We check the algebraic solutions geometrically as follows:

Example 1 *We consider the circle when $a_1 = 0, a_2 = 2, q = 1$, the fixed point $A = (0, a_2)$, and $s = 0, \frac{1}{2}, 1, \frac{3}{2}, -\frac{1}{2}$. The loci are depicted in red in Figure 2(a)-2(e) respectively using [4]. In addition, we include the corresponding GeoGebra file in [S1] for further explorations.*

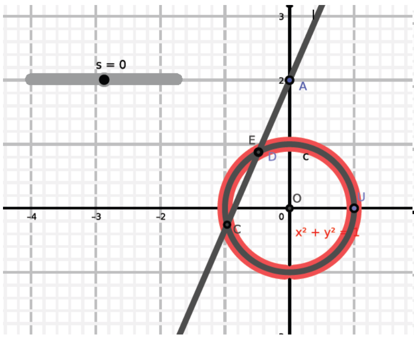

Figure 2(a). Locus when $s = 0$

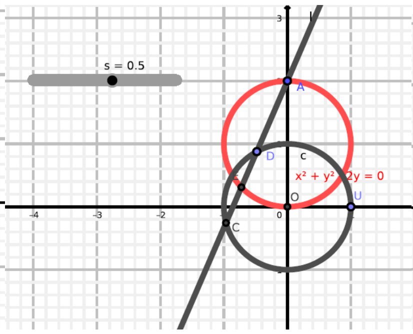

Figure 2(b). Locus when $s = 0.5$

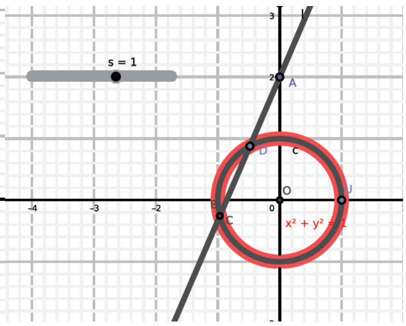

Figure 2(c). Locus when $s = 1$

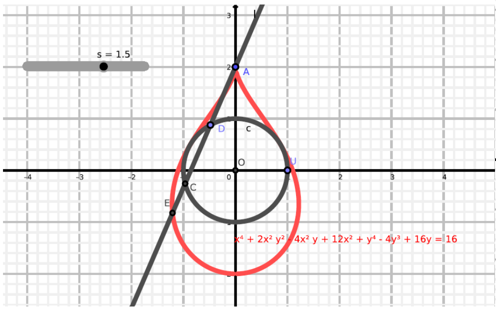

Figure 2(d). Locus when $s = 1.5$

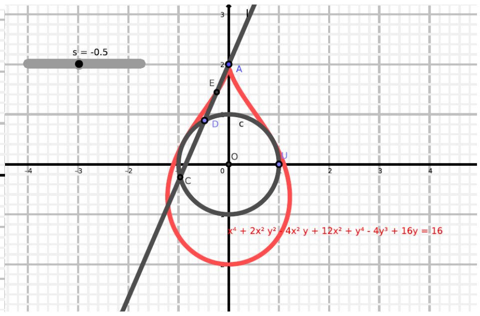

Figure 2(e). Locus when $s = -0.5$

3 Generalization to an ellipse and a special external point A

We consider the general ellipse of $x^2 + qy^2 = 1$, a quartic curve $\mathcal{C}(a_1 = 0, a_2, q > 0, s; x, y)$. We have the following equations:

$$
\begin{gathered}
e_1 := c_1^2 + qc_2^2 = 1, \\
e_2 := d_1^2 + qd_2^2 = 1, \\
e_3 := \det \begin{pmatrix} c_1 & c_2 & 1 \\ d_1 & d_2 & 1 \\ 0 & a_2 & 1 \end{pmatrix} = 0, \\
e_4 := x - d_1 = s \cdot (c_1 - d_1), \\
e_5 := y - d_2 = s \cdot (c_2 - d_2), \\
I := \langle e_1, e_2, \ldots, e_5 \rangle \cap \mathbb{Q}[a_2, s, x, y], \\
\langle p \rangle := I, \quad \mathcal{C} \mid p, \\
S := \{\mathcal{C} = 0, \tfrac{d\mathcal{C}}{dx} = 0, \tfrac{d\mathcal{C}}{dy} = 0\}.
\end{gathered}
$$

During an intermediate step we yield the following output of a principal ideal $I = \langle p \rangle$, which is an extension of the Equation (**??**).

$$
p = \left(qy^2 + x^2 - 1\right) \left(\begin{array}{c} q^2a_2^2y^2 + 4a_2^2qx^2s^2 - 4a_2^2qx^2s + a_2^2qx^2 - 4a_2^2qs^2 + 4a_2^2qs + a_2^2y^2 - a_2^2q \\ -2a_2qx^2y + 8a_2qs^2y - 8a_2qsy - 2a_2q^2y^3 + 2a_2qy \\ +x^4 - 4x^2s^2 + 4x^2s + 2qx^2y^2 - x^2 - 4qs^2y^2 + 4qsy^2 + qy^4 - qy^2 \end{array} \right).
$$

Similar to the circle case, the first factor corresponds to a degenerate case, where we identify the polynomial defining the ellipse. Therefore, we need only the second factor in the next step: `c:=(factor(p))[2]`.

Solution: Geometrically, only the solution $x = 0$ makes sense. Therefore, candidates for s are:

$$
0, 1, \frac{1 \pm a_2\sqrt{q}}{2}, \frac{1}{2}.
$$

3.1 Checking $s = \frac{1 \pm a_2\sqrt{q}}{2}$ geometrically

Example 2 *Consider the ellipse of $c : x^2 + qy^2 = 1$. If we set the fixed point $A = (0, a_2)$ with $a_2 = 2$ and $q = 4$, then $s = 2.5$ or $s = -1.5$ (which produces the same locus), is the boundary where the topology of the locus changes. We demonstrate such a scenario in Figure 3 below. We also include the GeoGebra file for readers to explore in [S2].*

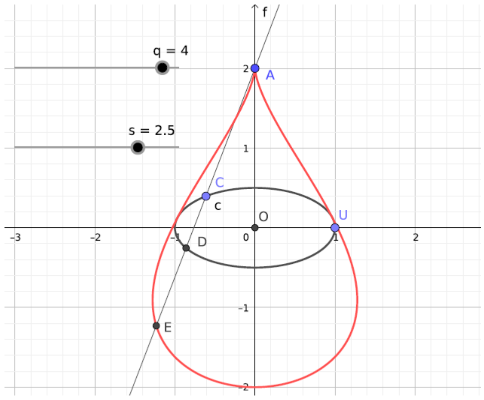

Figure 3. Locus changes its topology when $s = 2.5$ or -1.5

3.2 Full generalization

1. Computation of the Gröbner bases is too difficult (it leads to exceeding memory resources of more than 130 GB). We kindly thank RISC (Research Institute of Symbolic Computation, Hagenberg, Austria) for providing us with their "big memory server" to help us attempt computing the result in the general case.

2. We used Gröbner cover instead of traditional Gröbner basis computation, by using the computer algebra system **Singular** (See Appendix section for its input and output), and the package `grobcov`. See [15] for more details.

3. When performing the computation, there are lots of non-geometric and degenerate pieces of information. We searched for polynomials that contain variables a_1, a_2, q, s.

4. We factorized the polynomials and tried to find solutions for s that is a formula of a_1, a_2, q (but not of x or y).

5. With several attempts, we get a simple factorized form:

$$s \cdot (s-1) \cdot (2s-1)^2 \cdot (qa_2^2 + a_1^2 - 4s^2 + 4s - 1)^2 \cdot (y - a_2),$$

   and this is equal to 0 if, and only if, s is equal to $0, 1, \frac{1}{2}, \dfrac{1 \pm \sqrt{a_1^2 + qa_2^2}}{2}$ or $y = a_2$. (We are interested in s, so the last result does not play a role.)

Theorem 3 *(Algebraic Interpretation) If $a_1^2 + a_2^2 q - 1 \neq 0$ and $a_1^2 + a_2^2 q \neq 0$ and $3a_1^2 - a_2^2 q \neq 0$ and $q \neq 0$ and $a_2 \neq 0$ and $a_1 \neq 0$, then the locus curve $\mathcal{C}(a_1, a_2, q, s; x, y)$ may change its topology at $s = \dfrac{1 \pm \sqrt{a_1^2 + qa_2^2}}{2}$.*

Theorem 4 *(Geometric Interpretation) If $A \notin C$ and $a_1/a_2 \neq \sqrt{3}/q$ and A is outside of the axes, then the locus curve $\mathcal{C}(a_1, a_2, q, s; x, y)$ may change its topology at $s = \frac{1 \pm \sqrt{a_1^2 + qa_2^2}}{2}$.*

4 Is this still just a conjecture?

We actually obtained an algebraic statement. Its geometric counterpart should be verified! How can this be achieved for such a large number of input cases ($a_1, a_2 \in \mathbb{R}$, $q \in \mathbb{R}^+$)?

4.1 A proof based on affine transformations

1. Let us consider the general case $\mathcal{C}^*(a_1^*, a_2^*, q^* > 0, s; x, y)$, and we let $a_1 = a_1^*$, $a_2 = a_2^* \sqrt{q^*}$.
2. Let us stretch the whole construction to get $q = 1$.
   This results in a curve $\mathcal{C}\,(a_1, a_2, 1, s; x, y)$ because the dilation factor remains the same. The stretching ratios are $1 : \sqrt{q^*}$ with respect to $x : y$.
3. Since we have a circle now, it is allowed to rotate the whole construction about the origin to have $a_1' = 0$. So we obtain a second curve $\mathcal{C}'\,(0, a_2', 1, s; x, y)$ which can be studied via elimination without any difficulties.
4. Since we used stretching and rotation, the topology of the curves did not change.
5. By conclusion, we learn that the general case $\mathcal{C}(a_1^*, a_2^*, q^*, s; x, y)$ can be traced back to the case of a circle and a special point on the y-axis.

4.2 3D proof

By following the ideas from 2D:

1. We start with a fixed point $A = (0, a_2, 0)$ on the y-axis, and the ellipsoid whose equation is
   $$x^2 + qy^2 + qz^2 = 1, \tag{4}$$
   which is obtained by rotating the ellipse $x^2 + qy^2 = 1$ around the y-axis.
2. Using the same argument about the ellipse for $x^2 + qy^2 = 1$, as explained in Section 4.1, we prove that the topology for (4) changes at
   $$\frac{1 \pm a_2\sqrt{q}}{2}. \tag{5}$$
3. For the general case of the topological problem for the ellipsoid of the form
   $$x^2 + q_1 y^2 + q_2 z^2 = 1, \tag{6}$$
   we may apply an affine transformation, and the problem for Equation (6) can be traced back to the simple case of (4) and a special point on the y-axis.

4.3 Checking $s = \frac{1 \pm a_2\sqrt{q}}{2}$ geometrically for 3D

Example 5 *Consider the ellipsoid*

$$x^2 + qy^2 + qz^2 = 1, \tag{7}$$

with a fixed point $A = (0, a_2, 0)$ *on the* y*-axis. We set* $a_2 = 2$ *and* $q = 4$ *then it follows from the corresponding 2D formula (5) that* $s = 2.5$ *(or* $s = -1.5$*), is the boundary point where the locus changes its topology. We illustrate such scenarios in Figures 4(aa)-4(d) respectively when* $s = 2.25,\ 2.5, 2.7$ *and* 2.9.

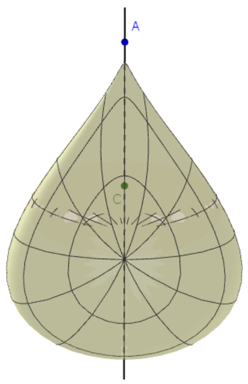

Figure 4(aa). Locus when $s = 2.25$

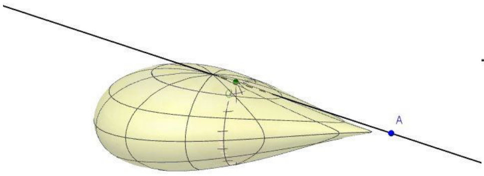

Figure 4(ab). Second view when $s = 2.25$

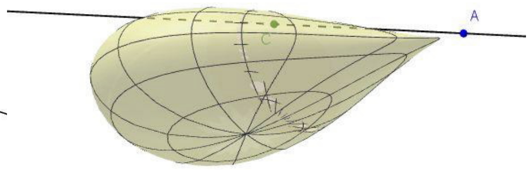

Figure 4(ac). Third view when $s = 2.25$

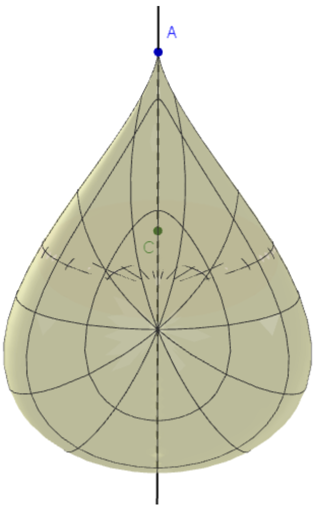

Figure 4(b). Locus when $s = 2.5$

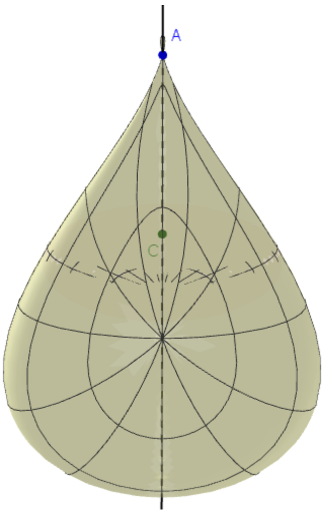

Figure 4(c). Locus when $s = 2.7$

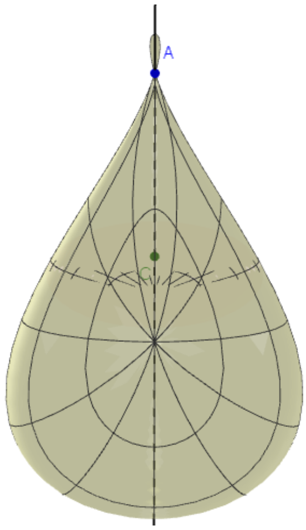

Figure 4(d). Locus when $s = 2.9$

5 Locus Surfaces and Linear Transformations

In earlier sections, we discussed how topological structures change with respect to the parameter s when the fixed point A is given at a finite distance from the origin. Now we summarize the scenario when the fixed point A is at an infinity, which has been discussed in [19]. For completeness, we summarize it here.

It is known that the image of an ellipsoid under a linear transformation is another ellipsoid. We proved this observation directly in [19] when the fixed point A is at an infinity. We quote two main results from [19] as follows:

Proposition 6 *If Σ is the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $A_{\text{inf}}(u_0, v_0)$ be the fixed point at an infinity in the direction of $(\cos u_0 \sin v_0, \sin u_0 \sin v_0, \cos v_0)$, $C \in \Sigma$ and D_{inf} be the "antipodal" point of C corresponding to $A_{\text{inf}}(u_0, v_0)$ satisfying $E_{\text{inf}} = sC + (1-s)D_{\text{inf}}$, then there exists a matrix*

$$L_E^e = sI + (1-s)L_D^e \tag{8}$$

such that $L_E^e C = E_{\text{inf}}$, and therefore, the locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ is the image of Σ under the linear transformation given by the matrix L_E^e.

Proposition 7 *For $s \in \mathbb{R} \setminus \{1\}$, the ellipsoid Σ and locus ellipsoid $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect themselves tangentially at an elliptical curve.*

Since the locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ is an ellipsoid and since $\Sigma \subset \Delta_{\text{inf}}(s, u_0, v_0)$ and two surfaces are tangent to each other. The locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ becomes a longer ellipsoid containing Σ as s increases. It is natural to imagine what will happen when $s \to \infty$. As one would expect that the locus ellipsoid $\Delta_{\text{inf}}(s, u_0, v_0)$ in this case would be blown up to a cylinder, which changes its topological structure, and it will not be topologically equivalent to an ellipsoid. We use the following example from [19] for demonstration.

Example 8 *We depict the locus elliptical cylinder (yellow) when $a = 5, b = 4, c = 3$, with $u_0 = \frac{\pi}{6}, v_0 = \frac{\pi}{3}$ together with the original ellipsoid (blue) in Figure 5(a). The two surfaces are tangent to an elliptical curve as seen in Figure 5(b).*

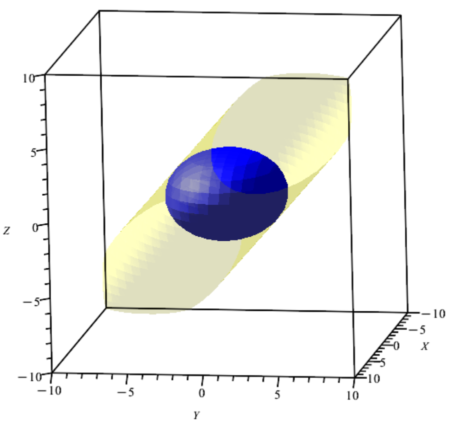

Figure 5(a). The original ellipsoid and its locus when $s \to \infty$.

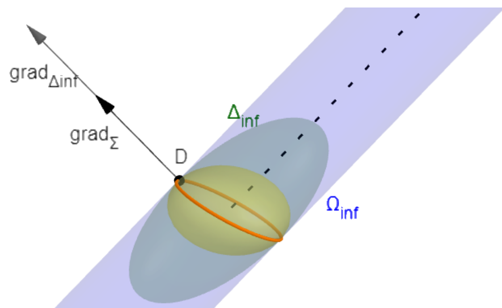

Figure 5(b). Ellipsoid and its locus are tangent at an elliptical curve..

6 Conclusion

The implementation of technological tools into research and teaching has changed the approach to old questions and also enabled the exploration of new situations, which until now seemed out of reach. Dynamic Geometry is the most popular tool among teachers and High School students, but some decades ago, software has been developed for more abstract domains, such as Number Theory, Group Theory, etc.. The traditional progression *definition-theorem-proof* with examples at some places has been replaced by *exploration-discovery-conjecture-proof*. Many mathematical domains became experimental.

Technology provide the researcher and the learner with crucial intuitions, before an attempt to find more rigorous analytical solutions. After experimentation with software, the question may be : "what happens here? what can I prove here?"

In this paper, we have gained geometric intuitions while using a DGS. College entrance exams can lead to challenging problems. Dynamic geometry software with computer algebra extensions may suggest valid conjectures. Most of the time, a DGS leads to discovery and conjecture, and a rigorous proof is obtained using the symbolic algebraic abilities of a CAS. The respective roles of DGS and CAS may be versatile, depending on the topic. Generally the DGS provides numerical support and the CAS symbolic exact work, but not always. GeoGebra Discovery has symbolic complements to numerical commands which pre-existed in GeoGebra. In [9], the CAS had to be used for numerical experimentation, not only the DGS. See also [8]. The working environment has changed, and the technological literacy is crucial. M. Artigue [1] points out that technological knowledge is an integral part of the new mathematical knowledge.

We extend systems with new functionalities (like symbolic `Dilation` for GeoGebra Discovery) that can speed up the investigation substantially. Free computer algebra systems (like Giac) can be comparable to commercial systems.

Evolving technological tools have made mathematics fun and accessible on the one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, more students can be attracted to the field, and inspired to investigate problems ranging from the simple to the more challenging.

Moreover, the usage of technology may encourage greater collaborations, between humans, and between man and machine. We give here also an example where different kinds of software have to collaborate with humans. Communication between machines is still to be developed. E. Roanes-Lozano [11] expressed often this wish, some progress has been made [13], but is still quite slow. Improving technological tools among researchers in mathematics and mathematics education, and developing all kinds of communication and collaboration, is a crucial task. Following Shulman [14], Loewenberg et al. [10] analyze and explain the importance of changing the teacher's knowledge for teaching. This requires also analyzing the didactic transition [7], from mathematical knowledge of taught knowledge ("du savoir savant" au "savoir enseigné"). We claim that working in the way we propose in this paper will not only spark new areas of research but also increase future teachers' content knowledge at the same time.

7 Supplementary Electronic Materials

[S1] GeoGebra file for Example 1: https://www.geogebra.org/m/fjb9bzz4.

[S2] GeoGebra file for Example 2: https://www.geogebra.org/m/xeykgv7t.

[S3] GeoGebra file for Example 5: https://www.geogebra.org/m/pkv54eeg.

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[2] Geometry Expressions, see http://www.geometryexpressions.com/.

[3] GeoGebra Discovery download: https://github.com/kovzol/geogebra-discovery.

[4] GeoGebra: See https://www.geogebra.org/.

[5] Giac/Xcas, a free computer algebra system: https://www-fourier.ujf-grenoble.fr/~parisse/giac.html.

[6] Maple, A product of Maplesoft, see http://Maplesoft.com/.

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8 Appendix

- Singular-input: LIB "grobcov.lib";

```
ring R=(0,a1,a2,q),(x,y,s),lp;
poly c=a2^2*q^2*y^2+4*a2^2*q*x^2*s^2-4*a2^2*q*x^2*s+a2^2*q*x^2
 -4*a2^2*q*s^2+4*a2^2*q*s-a2^2*q-2*a2*q^2*y^3-2*a2*q*x^2*y
 -8*a2*q*x*y*s^2*a1+8*a2*q*x*y*s*a1+8*a2*q*y*s^2-8*a2*q*y*s
 +2*a2*q*y+q^2*y^4+2*q*x^2*y^2-2*q*x*y^2*a1+4*q*y^2*s^2*a1^2
 -4*q*y^2*s^2-4*q*y^2*s*a1^2+4*q*y^2*s+q*y^2*a1^2-q*y^2
 +x^4-2*x^3*a1-4*x^2*s^2+4*x^2*s+x^2*a1^2-x^2+8*x*s^2*a1
 -8*x*s*a1+2*x*a1-4*s^2*a1^2+4*s*a1^2-a1^2;
poly dcx=4*a2^2*q*2*x*s^2-4*a2^2*q*2*x*s+a2^2*q*2*x-2*a2*q*2*x*y
 -8*a2*q*y*s^2*a1+8*a2*q*y*s*a1+2*q*2*x*y^2-2*q*y^2*a1+4*x^3
 -2*3*x^2*a1-4*2*x*s^2+4*2*x*s+2*x*a1^2-2*x+8*s^2*a1-8*s*a1+2*a1;
poly dcy=a2^2*q^2*2*y-2*a2*q^2*3*y^2-2*a2*q*x^2-8*a2*q*x*s^2*a1
 +8*a2*q*x*s*a1+8*a2*q*s^2-8*a2*q*s+2*a2*q+q^2*4*y^3+2*q*x^2*2*y
 -2*q*x*2*y*a1+4*q*2*y*s^2*a1^2-4*q*2*y*s^2-4*q*2*y*s*a1^2
+4*q*2*y*s+q*2*y*a1^2-q*2*y;
ideal I=c,dcx,dcy;
short=0;
grobcov(I); // the input is less than 900 characters
```

- Singular-output file can be found in this link:

https://ejmt.mathandtech.org/Contents/v19n1p1/singular-output.pdf.